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Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Real-Time Optimal Control of Aircraft Turn Trajectories

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Introduction

WIND speed and direction can be important factors in determining minimum-time turn trajectories for slow flying aircraft such as those involved in patrol, surveillance, and rescue. Frequently, the objective is to reach a nearby point in the shortest possible time. This is the so-called "undesignated fly-to-point problem"[1] (that is, the final approach heading is not specified). To provide automatic control for the execution of such maneuvers, the procedure for calculating the optimal trajectory must compute the solution in real time. Although this term will be made more specific later, it will generally mean here that the calculation must be completed before any appreciable change in aircraft position can occur and is usually measured in milliseconds of computation time on available computer equipment. This real-time solution is normally difficult to accomplish when the effect of wind on the ground track trajectory is included in the dynamics of the flight regime. An algorithm for determining such trajectories when the aircraft roll rate is not incorporated into the dynamics of the problem was given in Clements[2] and was based on simple geometrical arguments. Unfortunately, the relatively slow roll rates characteristic of large patrol aircraft can result in actual trajectories that vary considerably from these "unconstrained" computed solutions.

Although navigation programs based on the solution of singular optimal control problems have been studied for some years,[1-3] most of the work has focused on high-speed maneuvers where the effect of wind is not a consideration and the aircraft roll rate is not a dynamical constraint. The object of this work is to develop a mathematical formulation and a real-time solution procedure for the undesignated fly-to-point problem when the aircraft roll rates are incorporated into the dynamics of the problem. The results obtained here are based on the solution of a singular optimal control problem where the maximum performance roll rate and maximum performance bank angle determine the control and state variable constraints, respectively.

Definitions and Assumptions

The fly-to-point problem to be considered here is that of determining the minimum-time flight trajectory in the xy plane given the initial aircraft position (x_s, y_s), the fly-to-point position (x_f, y_f), the aircraft constant true airspeed in still air V, the initial aircraft heading angle $\Psi(0) = \Psi_s$, and the wind speed W from direction θ_W. The aircraft heading angle $\Psi(t)$ and the wind direction θ_W are in radians measured positive counterclockwise from the positive x axis. The aircraft bank angle or roll angle $\phi(t)$ is defined in the usual way as the angle in radians measured positive counterclockwise through which the vertical symmetry axis of the aircraft is rotated out of the plane defined by the longitudinal axis of the aircraft and the local vertical. The basic assumptions are the following:

1) The relevant (constant altitude) dynamics are defined in terms of a fixed x,y Cartesian coordinate system.
2) V is a fixed constant with $V > W$.
3) $\phi(t)$ is in $C[0, t_f]$ with $|\phi(t)| \leq \phi_M < \pi/2$ for some ϕ_M.
4) $\phi'(t)$ is in $C_S[0, t_f]$ with $|\phi'(t)| \leq \rho_M$ for some ρ_M.
5) $\Psi(t)$ is in $C^1[0, t_f]$ and satisfies $\Psi'(t) = (g/V)\tan\phi(t)$.
6) $\phi(0) = \phi(t_f) = 0$.

Here t_f is the total maneuvering time, g is the usual gravitational constant, and $C[0, t_f]$ and $C_S[0, t_f]$ are the linear spaces of continuous and piecewise (sectionally) continuous functions, respectively. The aircraft maximum performance constant rate of turn corresponds to the fixed angle of bank ϕ_M. The maximum roll rate of the aircraft is ρ_M. The stipulation $\phi(0) = \phi(t_f) = 0$ that all maneuvers begin and end in level flight is for simplicity only and does not affect the generality of the results obtained here. Assumption 5 has the effect of neglecting the inertial acceleration terms associated with the changes in aircraft heading and follows from the formula for the radius r of a balanced, standard-rate, coordinated turn at a fixed bank angle ϕ, $r = V^2/(g \tan\phi)$.

It follows from the assumptions that the dynamics of the problem can be formulated in terms of the system of equations

$$x'(t) = V\cos\Psi(t) - W\cos\theta_W$$

$$y'(t) = V\sin\Psi(t) - W\sin\theta_W$$

$$\Psi'(t) = (g/V)\tan\phi(t), \qquad \phi'(t) = \rho_M u(t) \tag{1}$$

for $0 \leq t \leq t_f$ and the end conditions

$t = 0$:

$$x(0) = x_s, \quad y(0) = y_s, \quad \phi(0) = 0, \quad \Psi(0) = \Psi_s \tag{2}$$

$t = t_f$:

$$x(t_f) = x_f, \quad y(t_f) = y_f, \quad \phi(t_f) = 0 \tag{3}$$

where $x(t)$, $y(t)$, and $\Psi(t)$ are in $C^1[0, t_f]$ and $u(t)$, $0 \leq t \leq t_f$, is identified as the primary control variable. Here $u(t) = 1$ corresponds to a maximum performance roll to the left and $u(t) = -1$ to a maximum performance roll to the right. The aircraft speed at any point $[x(t), y(t)]$ along the ground track trajectory is $V_G(t) = \{V^2 + W^2 - 2VW\cos[\Psi(t) - \theta_W]\}^{1/2}$.

Optimal Control $u^*(t)$, $0 \leq t \leq t_f$

The control and state variable constraints are $|u(t)| \leq 1$ and $S[\phi(t)] = [\phi(t) + \phi_M][\phi(t) - \phi_M] \leq 0$, respectively. It follows as

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in Maurer[4] that the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}(t) = 1 + \lambda_1(t)\left[V \cos \Psi(t) - W \cos \theta_W\right] + \lambda_2(t)\left[V \sin \Psi(t) - W \sin \theta_W\right] + \lambda_3(t)(g/V)\tan \phi(t) + \lambda_4(t)\rho_M u(t) + \eta(t) S\left[\phi(t)\right] \tag{4}$$

for Lagrange multipliers $\lambda_i(t)$, $i=1,\ldots,4$ satisfying the adjoint equations

$$\lambda_1'(t) = 0, \qquad \lambda_2'(t) = 0$$
$$\lambda_3'(t) = -\lambda_1(t)\, V \sin \Psi(t) + \lambda_2(t)\, V \cos \Psi(t)$$
$$\lambda_4'(t) = \lambda_3(t)(g/V)\sec^2\phi(t) - 2\eta(t)\phi(t) \tag{5}$$

for all t in $[0, t_f]$ where the multiplier $\eta(t) \geq 0$ satisfies $\eta(t)S[\phi(t)] \equiv 0$ on $[0, t_f]$. From the transversality conditions, $\lambda_3(t_f)=0$, and $\lambda_i(0)$, $\lambda_i(t_f)$, $i=1,2,4$, and $\lambda_3(0)$ are free parameters. Furthermore, since the final time is free and $\mathcal{H}(t)$ does not depend explicitly on t, $\mathcal{H}(t) \equiv 0$ for all t in $[0, t_f]$. A subarc of $\phi(t)$ for which $S[\phi(t)]<0$ is called an interior arc and a boundary arc is a subarc of $\phi(t)$ where $S[\phi(t)] \equiv 0$; $\eta(t)$ is in $C[0, t_f]$ and is continuously differentiable on the interior of any boundary arc.[4] This is a singular control problem with switching function $\sigma(t) = \partial\mathcal{H}/\partial u = \lambda_4(t)\rho_M$.

Since $\lambda_1'(t)=0$, $\lambda_2'(t)=0$ in Eq. (5), λ_1 and λ_2 are constant for all t in $[0, t_f]$. On any subinterval $(t_1, t_2]$ where $S[\phi(t)]<0$ and $\sigma(t)=0$, $\lambda_3'(t) = -\lambda_1 V \sin \Psi(t) + \lambda_2 V \cos \Psi(t) = 0$ and $\Psi(t) = \Psi^* = \arctan(\lambda_2/\lambda_1)$ if $\lambda_1 \neq 0$ or $\pm\pi/2$ if $\lambda_1=0$. Also, since $\mathcal{H}(t) \equiv 0$ for all t in $[0, t_f]$, Eq. (4) gives

$$1 + \lambda_1(V \cos \Psi^* - W \cos \theta_W) + \lambda_2(V \sin \Psi^* - W \sin \theta_W) = 0 \tag{6}$$

on $(t_1, t_2]$. Thus, $\phi(t)=0$ on $(t_1, t_2]$ and $u(t)=0$ is the singular control law on an interior arc. Moreover, if $\sigma(t)=0$ on $(t_1, t_f]$, this can be the only interior interval of singular control. For $S[\phi(t)]=0$ or $\phi(t)\phi_M$, $\phi'(t)=0$ and the (singular) boundary control is $u_b(t)=0$. Since the end conditions (2) and (3) include $\phi(0)=\phi(t_f)=0$, the time intervals where $u^*=1$ and $u^*=-1$ must be of equal total length.

Let $\tau_M = \phi_M/\rho_M$ denote the minimum time required to roll into or out of a maximum performance turn and let $T_M = (g \tan \phi_M)/(2\pi V)$ be the time required to complete a maximum performance 360-deg turn. Then the form of the optimal control $u^*(t)$ and the corresponding angle of bank history $\phi^*(t)$ must be given by

$$\left[u^*(t), \phi^*(t)\right] = \begin{cases} (\alpha, \rho^* t) & 0 \leq t < t_1 \\ (0, \rho^* t_1) & t_1 \leq t < t_2 \\ \left[-\alpha, \rho^*(t_1+t_2-t)\right] & t_2 \leq t < t_3 \\ \left[0, \rho^*(t_1+t_2-t_3)\right] & t_3 \leq t < t_4 \\ \left[\alpha, \rho^*(t_1+t_2-t_3+t_4+t)\right] & t_4 \leq t < t_5 \\ (0, 0) & t_5 \leq t \leq t_f \end{cases} \tag{7}$$

where

$$t_1 = \tau_M + sH(-s), \quad t_2 = \tau_M + sH(-s) + sH(s)$$
$$t_3 = 3\tau_M + 2sH(-s) + sH(s) + rH(-r)$$
$$t_4 = 3\tau_M + 2sH(-s) + sH(s) + rH(-r) + rH(r)$$
$$t_5 = 4\tau_M + 2sH(-s) + sH(s) + 2rH(-r) + rH(r) \tag{8}$$

for not necessarily distinct switching points t_i, $i=1,\ldots,5$. The s and r are real variables with $-\tau_M \leq s$, $r \leq T_M$, $\alpha = \pm 1$, $\rho^* = \alpha\rho_M$, and H is the usual Heaviside function. Here $s \geq 0$ corresponds to the time on the first boundary arc, $r \geq 0$ corresponds to the time on the second boundary arc, and $\alpha=1$ corresponds to an initial turn to the left whereas $\alpha=-1$ corresponds to an initial turn to the right.

Solution Procedure

To determine the required optimal trajectories, one would normally employ a multiple shooting method[4] to solve the two-point boundary-value problem defined by Eqs. (1-5) where the parameter estimates s, r, t_f, λ_1, and λ_2 are updated using Newton iteration. However, in what follows the simpler state-space solution procedure is employed because it does not require the direct calculation of the Lagrange multipliers. From Eqs. (1), (7), and (8), $\Psi^*(t)$, $0 \leq t \leq t_f$, is given by

$$\Psi^*(t) = \begin{cases} \beta^* \ln|\cos(\rho^* t)| + \Psi_s & 0 \leq t < t_1 \\ \Psi^*(t_1) - \beta^*\rho^*\left[\tan(\rho^* t_1)\right](t - t_1) & t_1 \leq t < t_2 \\ \Psi^*(t_2) + \beta\left\{\ln|\cos(\rho^* t_1)| - \ln|\cos[\rho^*(t_1+t_2-t)]|\right\} & t_2 \leq t < t_3 \\ \Psi^*(t_3) - \beta^*\rho^*\left\{\tan\left[\rho^*(t_1+t_2-t_3)\right]\right\}(t - t_3) & t_3 \leq t < t_4 \\ \Psi^*(t_4) + \beta^* \ln|\cos[\rho^*(t-t_4)]| & t_4 \leq t < t_5 \\ \Psi^*(t_5) & t_5 \leq t \leq t_f \end{cases} \tag{9}$$

with $\beta^* = -g/(V\rho^*)$ and the optimal trajectory $[x^*(t), y^*(t)]$, $0 \leq t \leq t_f$, can be recovered from Eqs. (8) and (9) by direct integration of Eq. (1). It follows from Eq. (6) with $(t_1, t_2) = (t_5, t_f)$ that, for each fixed s in Eq. (8), r must be a solution of the fixed point equation.

$$G\left[r(s)\right] = x'(t_5)\left[y(t_5) - y_f\right] - y'(t_5)\left[x(t_5) - x_f\right] = 0 \tag{10}$$

where $x(t_5)$ and $y(t_5)$ are obtained from Eq. (11). For each s, $-\tau_M \leq s \leq T_M$, let $F[s, r(s)]$ be defined by

$$F\left[s, r(s)\right] = t_5 + \left\{\left[x(t_5) - x_f\right]^2 + \left[y(t_5) - y_f\right]^2\right\}^{1/2} / V_G(t_5) = t_f \tag{11}$$

where $r(s)$ is given by Eq. (10). Then the required switching points t_i, $i=1,\ldots,5$, can be determined by minimizing the function $F[s, r(s)]$ in Eq. (11) once for $\alpha=1$ and once for $\alpha=-1$. The optimal trajectory will be defined by the minimum of these two feasible extremals. A Fortran 77 computer program was written to carry out the solution procedure described in this section, and an example application is considered next. The results given here form the basis for one component of a general steering program currently in operation in a Canadian Forces patrol aircraft.[1]

Example Application

In what follows, distances and positions are given in nautical miles from a fixed origin in the xy plane where true north is defined by the direction of the positive y axis. Time is measured in seconds, speed is in knots, ϕ_M is taken to be $\pi/6$, and the switching times t_i^*, $i=1,\ldots,5$, and t_f^* are computed to within an error tolerance of 0.1 s. The ϕ_M represents a turn at 30 deg angle of bank, and ρ_M is determined by the minimum time required to roll from $\phi=0$ to ϕ_M, which here is taken to be 8 s. It is assumed that the aircraft has equipment to monitor its own position and that of the fly-to-point. The specific case considered corresponds to a fly-to-point 1 n.mi. off the left

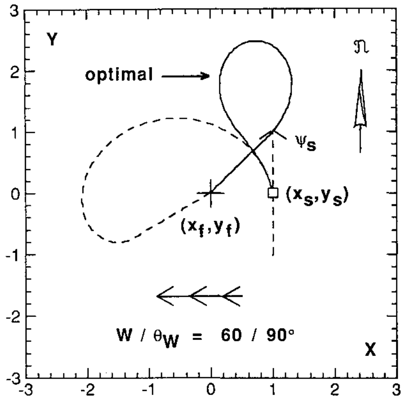

Fig. 1 $[x^*(t), y^*(t)]$, $0 \le t \le t_f$.

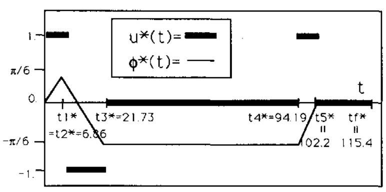

Fig. 2 $u^*(t)$, $\phi^*(t)$, $0 \le t \le t_f$, $(\alpha^* = 1)$.

wing tip of an aircraft flying due north at 200 kt with a 60-kt crosswind from the right:

Example 1:

$$(x_s, y_s), \quad (x_f, y_f), \quad V\,(\text{kt}), \quad \Psi_s, \quad W\,(\text{kt}), \quad \theta_W$$

Input:

$$(1, 0), \quad (0, 0), \quad 200, \quad \pi/2, \quad 60, \quad 0$$

The minimum-time trajectory for this fly-to-point objective was computed to be $\alpha^* = 1$, $t_1^* = t_2^* = 6.86$, $t_3^* = 21.73$, $t_4^* = 94.19$, $t_5^* = 102.2$, and $t_f^* = 115.4$, which is a left-right turn combination requiring 115.4 s total flight time. The optimal trajectory is given by the solid line in Fig. 1. The optimal control profile $u^*(t)$ and corresponding angle of bank history $\phi^*(t)$ are given in Fig. 2. The total clock time on a Macintosh IIci microcomputer required to compute the solution for this specific wind and fly-to-point configuration was 233 ms. Although machine language code and faster computer equipment is used in actual practice, this is already well within the time scale required for the automatic control of the maneuver. What is interesting about this example is that the computed optimal trajectory consisting of a left-right turn combination is not one that an aircrew would normally execute. Indeed, experience has shown[1] that, without additional information, the pilot in command will either choose the nonoptimal feasible solution (the dashed line in Fig. 2 that requires 136.7 s flight time) or will first execute a procedure turn into wind and away from either feasible extremal. There are many similar scenarios where the optimal trajectory is equally unintuitive.

Conclusion

In the problem considered here, an analysis of the standard necessary conditions leads to the required form for the optimal control and an efficient state-space method of solution. An important advantage of this real-time solution procedure is that the optimal trajectories can be recomputed throughout the execution of the maneuver to allow for the correction of cross-track errors (unexpected drift) and inaccurate estimates of wind speed and direction.

Acknowledgments

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U-Parameter Design for Terrain-Following Flight Control

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Introduction

THIS Note explores the application of U-parameter feedback design developed by Dorato and Li[1] to a terrain-following flight control problem. U-parameter theory permits one to optimize performance of a nominal linear time invariant system, while guaranteeing robust stability in the presence of unstructured plant perturbations. In the terrain-following problem considered here, it is assumed that a command flight path angle is computable from available radar data (see, for example, Ref. 2) so as to achieve good terrain following and that the object is to find a controller that causes the actual flight path angle to follow the command signal. The control input is taken to be the elevator angle, and the aircraft dynamics are linearized about constant nominal trajectories. Since the most significant variation in the linearized model is due to nominal flight path angle, this variable is treated as an uncertain parameter. This parameter uncertainty is transferred to an unstructured frequency domain bound on plant transfer function uncertainty. U-parameter theory is then used to minimize a norm on tracking error for nominal motion (level flight 100 m above terrain and zero flight path angle), while guaranteeing robust stability for other flight path angles. Before discussing the terrain-following problem, we briefly review U-parameter theory.

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